Finite Element Modeling with ANSYS

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Course Overview

❖ *Objective*: Understand course expectations and policies.

"An FEA program allows an engineer to make mistakes at a rapid rate of speed. (R. Miller)"

This is important, because a popular tenet of current design thought suggests that you should fail often and fail cheaply—in other words, bring your model in contact with reality as often as you can sustain in order to exploit the corrections that come from

To a certain extent, I want you to treat everything I suggest in this class as a set of guidelines which can be explored and challenged. Think of the computational environment as itself empirical—it is often easy for you to test small hypotheses and craft working problems which can be compared to each other and to reality. Many questions in the course may thus be answered by the rejoinder—*do it!*

A Brief Review of the Finite Element Method

❖ *Objective*: Understand the underlying computational process of the finite element method.

Solution Process

The basic algorithm for a finite element solution is as follows:

1. Decompose the elements into structures and write a governing equation relationship (such as force-displacement) for each element.
2. Assemble all of the elements to get the governing equation statement for the entire problem, $\underline{\underline{K}}\,\underline{c} = \underline{F}$K__c_=F_.
3. Apply the boundary conditions, $\underline{\underline{K}}^*\underline{c} = \underline{F}^*$K__∗c_=F_∗.
4. Solve the resulting linear system to get the solution $\underline{c}$c_.
5. Post-process this result to extract stress, strain, displacement, and other variables of interest.

Mathematically, we proceed a bit differently, but it is easier to see where the foregoing steps arise if you have an understanding of the four basic forms of the problem: strong $\mathbb{S}$, weak $\mathbb{W}$, Galerkin $\mathbb{G}$, and matrix $\mathbb{M}$.

$\mathbb{S}$ Strong Form

The strong form is the canonical mathematical statement of the boundary-value problem which you are accustomed to from your previous engineering studies. The governing equations are stated for the domain, along with boundary conditions.

Given f, g, h, find $u(x)$ such that

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + f = 0 \qquad 01$$
$$u(1) = g$$
$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = -h$$

d2u(x)dx2+f=001u(1)=gdudx|x=0=−h

$\mathbb{W}$ Weak Form

The weak form is an equivalent variational statement of the problem. A solution of $\mathbb{S}$ satisfies $\mathbb{W}$, and vice versa.

Integrate by parts to construct the weak form. The weighting functions$w(x)$ are arbitrary as long as they satisfy the homogeneous form of the essential boundary conditions: $w(x = 1) = 0$. Then the weak statement of the problem is:

For any $w(x)$ such that $w(x = 1) = 0$, find $u(x)$ such that the following equation is satisfied:

$$\int_0^1 \mathrm{d}x \, \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} = w(x)\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2}\Bigg|_{x=0}^{1} - \int_0^1 \mathrm{d}x \, \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} w(x)$$

$$= -w(x)f(x)\Big|_{x=0}^{1} + \int_0^1 \mathrm{d}x \, f(x)w(x) \qquad \text{by } \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + f = 0$$

$$\int_0^1 \mathrm{d}x \, \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} = \int_0^1 \mathrm{d}x \, w(x)f(x) + hw(0)$$

∫01dxdwdxdudx=w(x)d2u(x)dx2|x=01−∫01dxd2u(x)dx2w(x)=−w(x)f(x)|x=01+∫01dxf(x)w(x)byd2u(x)dx2+f=0

$\mathbb{G}$ Galerkin Form

Although the matrix form $\mathbb{M}$ is the actual statement of the problem which is solved by the machine, it is the Galerkin form $\mathbb{G}$ which dominates the numerics. At this point, we assume a discretization (a mesh) and make approximations as to how we will represent the components of the solution.

Discretize the domain and approximate u and w by $\tilde{u}$u~ and $\tilde{w}$w~.

For any $\tilde{w}(x)$w~(x) such that $\tilde{w}(x = 1) = 0$w~(x=1)=0, find $\tilde{u}(x)$u~(x) such that the following equation is satisfied:

$$\int_0^1 \mathrm{d}x \, \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x} = \int_0^1 \mathrm{d}x \, w(x)f(x) + hw(0) \Rightarrow$$

$$\int_0^1 \mathrm{d}x \, \frac{\mathrm{d}\tilde{w}}{\mathrm{d}x}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}x} = \int_0^1 \mathrm{d}x \, \tilde{w}(x)f(x) + h\tilde{w}(0)$$

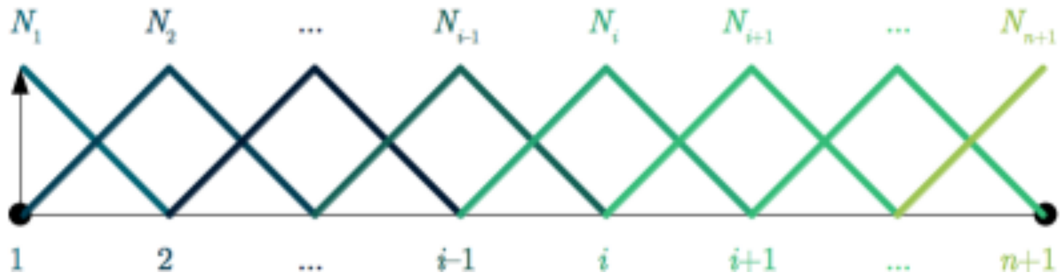
∫01dxdwdxdudx=∫01dxw(x)f(x)+hw(0)⇒∫01dxdw~dxdu~dx=∫01dxw~(x)f(x)+hw~(0)

$\mathbb{M}$ Matrix Form

Although the Galerkin form is approximate, we have not yet specified the discretized approximating functions $\tilde{u}$u~ and $\tilde{w}$w~. We will treat these as composed of shape functions N_i, which are equal to zero outside of the interval $x_{i-1} \leq x \leq x_{i+1}$. The form of the shape functions can be varied, but we will here use a linear form:

$$N_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{h_i} & x_i \leq x \leq x_{i+1} \\ 0 & \text{elsewhere} \end{cases} .$$

Ni(x)={x−xi−1hixi−1≤x≤xixi+1−xhixi≤x≤xi+10elsewhere.



The given approximating functions use these as a basis:

$$\begin{aligned}\tilde{u}(x) &= \sum_{i=1}^{n+1} c_i N_i(x)\\ &= \sum_{i=1}^{n} c_i N_i(x) + g N_{n+1}(x)\end{aligned}$$

$$\begin{aligned}\tilde{w}(x) &= \sum_{i=1}^{n+1} d_i N_i(x)\\ &= \sum_{i=1}^{n} d_i N_i(x)\end{aligned}$$

u~(x)=∑i=1n+1ciNi(x)=∑i=1nciNi(x)+gNn+1(x)w~(x)=∑i=1n+1diNi(x)=∑i=1ndiNi(x)

where c_i, d_i are weighting coefficients.

With these definitions, we are equipped to rewrite the Galerkin form in an equivalent matrix form. Substitute for $\tilde{u}$u~ and $\tilde{w}$w~ in $\mathbb{G}$:

$$\int_0^1 \mathrm{d}x \left[\left(\sum_{i=1}^{n} d_i \frac{\mathrm{d}N_i}{\mathrm{d}x}\right)\left(\sum_{j=1}^{n} c_j \frac{\mathrm{d}N_j}{\mathrm{d}x} + g\frac{\mathrm{d}N_{n+1}}{\mathrm{d}x}\right)\right] = \int_0^1 \mathrm{d}x \left[\left(\sum_{i=1}^{n} d_i N_i(x)\right) f\right] + \sum_{i=1}^{n} d_i N_i(0)$$

∫01dx[(∑i=1ndidNidx)(∑j=1ncjdNjdx+gdNn+1dx)]=∫01dx[(∑i=1ndiNi(x))f]+∑i=1ndiNi(0)

Rewrite this equation as $\sum_{i=1}^{n} d_i G_i(x) = 0$∑i=1ndiGi(x)=0, where

$$G_i = \sum_{j=1}^{n}\left[\int_0^1 \mathrm{d}x\, c_j \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_j}{\mathrm{d}x}\right] + g\left[\int_0^1 \mathrm{d}x \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_{n+1}}{\mathrm{d}x}\right] - \int_0^1 \mathrm{d}x\, N_i(x) f - h N_i(0).$$

Gi=∑j=1n[∫01dxcjdNidxdNjdx]+g[∫01dxdNidxdNn+1dx]−∫01dxNi(x)f−hNi(0).

Define

$$\begin{aligned}K_{ij} &\equiv \int_0^1 \mathrm{d}x \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_j}{\mathrm{d}x} && i,j \in \{1,\ldots,n\}\\ F_i &\equiv \int_0^1 \mathrm{d}x\, N_i(x) f + h N_i(0) - g\left[\int_0^1 \mathrm{d}x \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_{n+1}}{\mathrm{d}x}\right] && i \in \{1,\ldots,n\}\,.\end{aligned}$$

Kij≡∫01dxdNidxdNjdxi,j∈{1,...,n}Fi≡∫01dxNi(x)f+hNi(0)−g[∫01dxdNidxdNn+1dx]i∈{1,...,n}.

We can now write the matrix form of the problem, which will be solved using conventional linear-algebraic methods.

Given the coefficient matrix $\underline{\underline{K}}$K__ and vector $\underline{F}$F_, find $\underline{c}$c_ such that

$$\underline{\underline{K}}\,\underline{c} = \underline{F}.$$

K__c_=F_.

Solution Properties

A solution to either the strong form $\mathbb{S}$ or the weak form $\mathbb{W}$ will satisfy the other; similarly, solutions to either of the Galerkin form $\mathbb{G}$ or the matrix form $\mathbb{M}$ (defined over the same basis functions) will satisfy both. The primary approximation is the transformation from the weak formulation $\mathbb{W}$ to the approximate Galerkin form $\mathbb{G}$.

$$\mathbb{S} \leftrightarrow \mathbb{W} \leftrightsquigarrow \mathbb{G} \leftrightarrow \mathbb{M}$$

S↔W↭G↔M

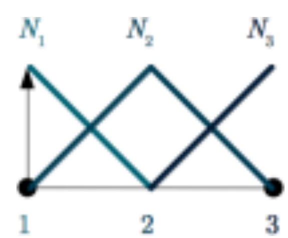
The stiffness matrix $\underline{\underline{K}}$K__ is symmetric, banded, and positive-definite, making it attractive for conventional matrix solution methods.

Example Solutions

Consider the foregoing system with three nodes and two elements. We wish to solve the system by hand for a few values of *f*, *g*, *h*.

Grid and Shape Functions

The shape functions are a function of the nodes (here, located equidistant at $x = \left\{0, \frac{1}{2}, 1\right\}$x={0,12,1}), and not of the equation.



$$N_1(x) = \begin{cases} \frac{x-0}{\frac{1}{2}} \to \quad 1-2x & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases}$$

N1(x)={x−012→1−2x0≤x≤12012≤x≤1

$$N_2(x) = \begin{cases} \frac{\frac{1}{2}-x}{\frac{1}{2}} \to \quad 2x & 0 \le x \le \frac{1}{2} \\ \frac{1-x}{\frac{1}{2}} \to \quad 2-2x & \frac{1}{2} \le x \le 1 \end{cases}$$

N2(x)={12−x12→2x0≤x≤121−x12→2−2x12≤x≤1

$$N_3(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ \frac{x-1}{\frac{1}{2}} \to \quad 2x-2 & \frac{1}{2} \le x \le 1 \end{cases}.$$

N3(x)={00≤x≤12x−112→2x−212≤x≤1.

Example 1: $f = 0, g = 0, h = 1, n = 2$

The problem statement reduces to

Find $u(x)$ such that

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = 0 \qquad 0 < x < 1$$
$$u(1) = 0$$
$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = -1$$

d2u(x)dx2=00<x<1u(1)=0dudx|x=0=−1

To find the resulting matrix formulation $\mathbb{M}$:

The matrix elements of $\underline{\underline{K}}$K__ are

$$K_{11} = \int_0^1 \mathrm{d}x \frac{\mathrm{d}N_1}{\mathrm{d}x}\frac{\mathrm{d}N_1}{\mathrm{d}x} = \int_0^{\frac{1}{2}} (-2)(-2) = 4x\Big|_{x=0}^{\frac{1}{2}} = 2$$

$$K_{12} = K_{21} = \int_0^1 \mathrm{d}x \frac{\mathrm{d}N_1}{\mathrm{d}x}\frac{\mathrm{d}N_2}{\mathrm{d}x} = \int_0^{\frac{1}{2}} (2)(-2) = -4x\Big|_{x=0}^{\frac{1}{2}} = -2$$

$$K_{22} = \int_0^1 \mathrm{d}x \frac{\mathrm{d}N_2}{\mathrm{d}x}\frac{\mathrm{d}N_2}{\mathrm{d}x} = \int_0^{\frac{1}{2}} (2)(2) + \int_{\frac{1}{2}}^1 (-2)(-2) = -4x\Big|_{x=0}^{\frac{1}{2}} + 4x\Big|_{x=\frac{1}{2}}^{1} = 4$$

K11=∫01dxdN1dxdN1dx=∫012(−2)(−2)=4x|x=012=2K12=K21=∫01dxdN1dxdN2dx=∫012(2)(−2)=−4x|x=012=−2K2

thus $\underline{\underline{K}} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}$K__=[2−2−24].

The solution vector $\underline{F}$F_ is

$$F_1 = \frac{1}{2}N_1(0) = \frac{1}{2}$$
$$F_2 = \frac{1}{2}N_2(0) = 0$$

F1=12N1(0)=12F2=12N2(0)=0

and $\underline{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$F_=[10].

The unknown vector $\underline{c}$c_ is now found as

$$\underline{\underline{K}}\,\underline{c} = \underline{F} \Rightarrow$$
$$\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \underline{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$
$$\underline{c} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\underline{c} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

K__c_=F_⇒[2−2−24]c_=[10]⇒c_=[2−2−24]−1×[10]c_=[112].

Substituting the coefficients back into the approximating solution $\tilde{u}$u~ now yields the FEM solution:

$$\tilde{u}(x) = \sum_{i=1}^{n} c_i N_i(x) + g N_{n+1}(x) = 1 \begin{cases} 1 - 2x \\ 0 \end{cases} + \frac{1}{2} \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \le x \le 1 \end{cases} = \begin{cases} 1 - 2x + x \\ 1 - x \end{cases} = 1 - x$$
u~(x)=∑i=1nciNi(x)+gNn+1(x)=1{1−2x0+12{2x0≤x≤122−2x12≤x≤1={1−2x+x1−x=1−x

The exact solution is $u = 1 - x$, in this case corresponding exactly to the FEM solution.

Example 2: $f \in \mathbb{R}$, $g = 0$, $h = 1$, $n = 2$

For arbitrary real f, the problem statement becomes

Find $u(x)$ such that

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + f = 0 \qquad 0 < x < 1$$
$$u(1) = 0$$
$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = -1$$
d2u(x)dx2+f=00<x<1u(1)=0dudx|x=0=−1

The nodes have not changed, so the shape functions remain the same. As before, we find the components of the matrix solution.

The matrix elements of the stiffness matrix $\underline{\underline{K}}$ K__ remain the same, as they are not a function of the source term f.

The solution vector $\underline{F}$ F_ becomes

$$\begin{aligned} F_1 &= \int_0^1 \mathrm{d}x\, N_1(x) f + 1 \cdot N_1(0) \\ &= \int_0^{\frac{1}{2}} \mathrm{d}x\, (1 - 2x) f + 1 = 1 + \frac{1}{4} f \\ F_2 &= \int_0^1 \mathrm{d}x\, N_2(x) f + 1 \cdot N_2(0) \\ &= \int_0^{\frac{1}{2}} \mathrm{d}x\, (2x) f + \int_{\frac{1}{2}}^1 \mathrm{d}x\, (2 - 2x) f = \frac{1}{2} f \end{aligned}$$
F1=∫01dxN1(x)f+1⋅N1(0)=∫012dx(1−2x)f+1=1+14fF2=∫01dxN2(x)f+1⋅N2(0)=∫012dx(2x)f+∫121dx(2−2x)f=12f

and $\underline{F} = \begin{bmatrix} 1 + \frac{1}{4} f \\ \frac{1}{2} f \end{bmatrix}$ F_=[1+14f12f].

The unknown vector $\underline{c}$ c_ is now found as

$$\underline{\underline{K}}\,\underline{c} = \underline{F} \Rightarrow$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \underline{c} = \begin{bmatrix} 1 + \frac{1}{4}f \\ \frac{1}{2}f \end{bmatrix} \Rightarrow$$

$$\underline{c} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}^{-1} \times \begin{bmatrix} 1 + \frac{1}{4}f \\ \frac{1}{2}f \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \times \begin{bmatrix} 1 + \frac{1}{4}f \\ \frac{1}{2}f \end{bmatrix}$$

$$\underline{c} = \begin{bmatrix} 1 + \frac{1}{2}f \\ \frac{1}{2} + \frac{3}{8}f \end{bmatrix}.$$

K__c_=F_⇒[2−2−24]c_=[1+14f12f]⇒c_=[2−2−24]−1×[1+14f12f]=[1121212]×[1+14f12f]c_=[1+12f12+38f].

Substituting the coefficients back into the approximating solution $\tilde{u}$ now yields the general FEM solution for the case of $f \in \mathbb{R}$, $g = 0$, $h = 1$, $n = 2$:

$$\tilde{u}(x) = \sum_{i=1}^{n} c_i N_i(x) + g N_{n+1}(x) = \left(1 + \frac{1}{2}f\right) \begin{cases} 1 - 2x \\ 0 \end{cases} + \left(\frac{1}{2} + \frac{3}{8}f\right) \begin{cases} 2x \\ 2 - 2x \end{cases} = \begin{cases} \left(1 + \frac{1}{2}f\right) + \left(-1 - \frac{1}{4}f\right) x \\ \left(1 + \frac{3}{4}f\right) + \left(-1 - \frac{3}{4}f\right) x \end{cases}$$

u~(x)=∑i=1nciNi(x)+gNn+1(x)=(1+12f){1−2x0+(12+38f){2x2−2x={(1+12f)+(−1−14f)x0≤x≤12(1+34f)+(−1−34f)

The exact solution is quadratic:

$$u(x) = -\frac{f}{2}x^2 - x + \left(1 + \frac{f}{2}\right).$$

u(x)=−f2x2−x+(1+f2).

The FEM solution approximates this solution along the element intervals and matches the exact solution at the nodes. Some specific cases are plotted in the figure below. (The black grid represents the approximate solution $\tilde{u}(x)$, and the smooth colored surface represents the real solution $u(x)$.)