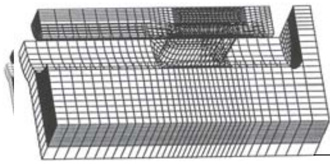

ME498 ABAQUS: A short course for engineers

Finite Element Stress Analysis: Theory; failure criteria; Example Application

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What is a process model?

Input Data → Equations and Constants → Output Data

- Makes quantitative predictions about process

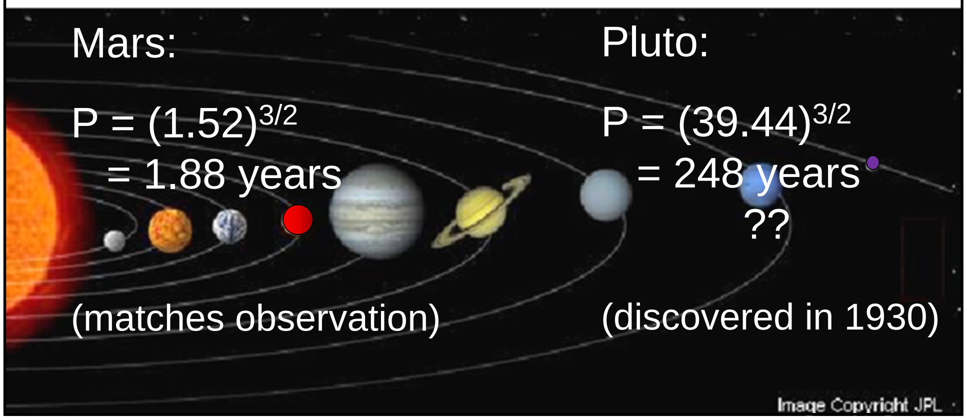
See supplement: BG Thomas & JK Brimacombe, "Process Modeling", 1997, 16 pages.

Example Model:

How long is a planet's orbit (years)

Kepler's 3rd Law of Planetary Motion, 1619

Distance from Sun (in Earth Orbital Radii) → $P = R^{3/2}$ → Orbit Period (in Earth years)

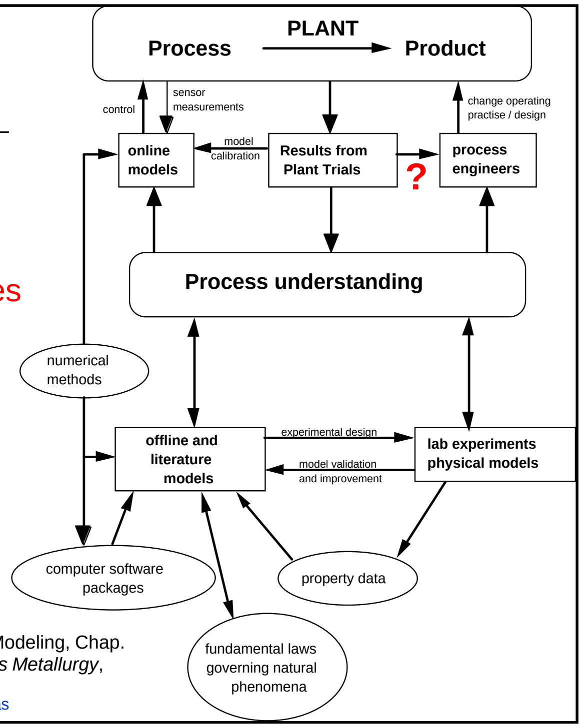


Steps in Modeling - summary

1. Define the real-world problem and model (including constants)
2. Calibrate model to match known data (use dimensionless numbers) - Earth
3. Validate the model - Mars
   – compare with known solution
4. Use the model
   • learn something new! - Pluto
5. Extend the model further (needs to be more fundamental) - other solar system

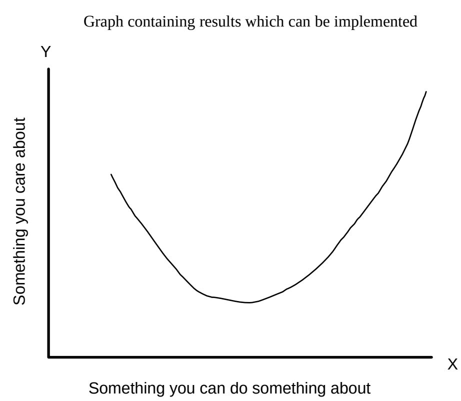
What is model Implementation?

inducing beneficial changes to process operation



Thomas, B. and J.K. Brimacombe, Process Modeling, Chap. 8, in *Advanced Physical Chemistry in Process Metallurgy*, 1997

Literature Models



Why Model?

- increase fundamental understanding
- technology transfer
- design of experiments
- evaluation of alternative designs
- process optimization
- extension and evaluation of plant results
- extending lab measurements to quantify properties
- assist in scale-up
- online process control

If a clear reason to develop a model cannot be found, then it should not be developed!

Find FEM Equations
1) Review Governing Eqns.-3D Elasticity

- Unknown field variables to solve for:
  - 3 displacements: u,v,w
  - 6 strains: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}$
  - 6 stresses: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}$
- Equations to solve:
  - 3 equilibrium equations: $[\Lambda]^T \{\sigma\} = \{F\}$
  - 6 constitutive equations: $\{\sigma\} = [D]\{\varepsilon\}$
  - 6 strain-displacement eqs.: $\{\varepsilon\} = [\Lambda]\{u\}$

Find FEM Eqs.: Galerkin's Method

- Substitute:

$$\underset{6\times1}{\{\sigma\}} = [D]\{\varepsilon\} = \underset{6\times6}{[D]}\underset{6\times3}{[\Lambda]}\begin{Bmatrix}u\\v\\w\end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix}u\\v\\w\end{Bmatrix} = \underset{3\times n}{[N]}\underset{n\times1}{\{d\}}$$

shape functions

$$\int_V ([\Lambda][N])^T[D]([\Lambda][N])dV\ \{d\} = \int_S \left([N]^T\{\sigma\}\right)dS + \int_V [N]^T\{F\}dV$$

$$\underbrace{\int_V \left(\underset{\substack{n\times6\\n\times3}}{[B]}\right)^T \underset{\substack{6\times6\\3\times3}}{[D]}\left(\underset{\substack{6\times n\\3\times n}}{[B]}\right)dV}_{\underset{n\times n}{[K]}} \underset{n\times1}{\{d\}} = \underbrace{\int_V [N]^T\{F\}dV}_{\text{body forces}} + \underbrace{\int_S\left([N]^T\{\sigma\}\right)dS}_{\text{boundary conditions}}$$

3D
2D

boundary conditions:
surface tractions
pressures
point loads

$[K]$ units $\left(\frac{1}{m}\right)\left(\frac{N}{m^2}\right)\left(\frac{1}{m}\right) m^3 = \frac{N}{m}$ $\qquad \left(\frac{N}{m^3}\right)m^3 = N$

Global Equation System

$$\underset{\frac{N}{m}}{[K]}\underset{m}{\{d\}} = \{F\}_{bf} + \{F\}_{\Phi} + \{F\}_{\varepsilon_o} + \underset{N}{\{P\}}$$

- Force Vectors

$$\{F\}_{bf} = \int_V [N]^T\{F\}dV$$ body forces $\left(\frac{N}{m^3}\right)(m^3) = N$

$$\{F\}_{\Phi} = \int_S [N]^T\{\Phi\}dS$$ surface tractions $\left(\frac{N}{m^2}\right)(m^2) = N$

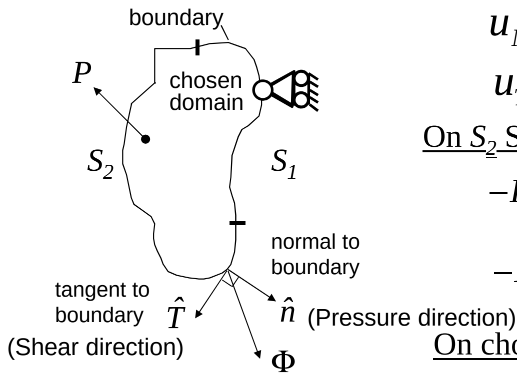
$$\{F\}_{\varepsilon_0} = \int_V [B]^T[D]\{\varepsilon_0\}dV$$ Initial strain forces $\quad \sigma = [D]\underbrace{\left(\{\varepsilon\} - \{\varepsilon_0\}\right)}_{\{\varepsilon_{elastic}\}}$

$\{P\} =$ point loads (N)

$\{\varepsilon_0\}$ = "initial strain": plastic, creep, thermal, etc.

Stress Boundary Conditions

- Specify 1 of the following on each portion of the domain boundary for each DOF (2 DOF for 2-D problems and 3 for 3-D):



On S_1 Specify Displacement(s)

$$u_N = \bar{u}_N(x,y,z)$$

$$u_T = \bar{u}_T(x,y,z)$$

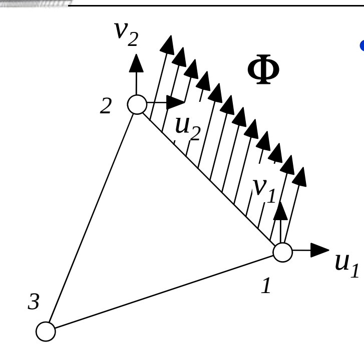
On S_2 Specify Surface Traction(s)

$$-E\frac{\partial u_N}{\partial n} = \Phi_{Pressure\ traction}$$

$$-E\frac{\partial u_T}{\partial n} = \Phi_{Shear\ traction}$$

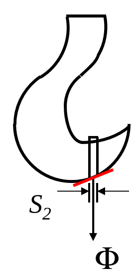
On chosen nodes: Point Load(s)

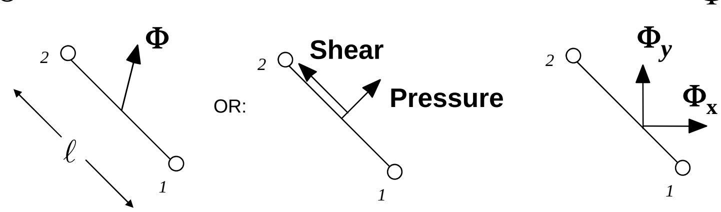
$\{F_\Phi\}$ Surface Traction Force Vector on Side of a CST



- Apply traction $\Phi\,(N/m^2)$ on side 1-2

example





Stress Analysis Output

- 6 strain components (isotropic)
  - ε_x, ε_y, ε_z, (normal) γ_{xy}, γ_{yz}, γ_{zx} (shear)

From u,v,w:

$$\{\varepsilon\} = [B]\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

- 6 stress components (isotropic)
  - σ_x, σ_y, σ_z, (normal) τ_{xy}, τ_{yz}, τ_{zx} (shear)

$$\{\sigma\} = [D]\{\varepsilon\}$$

- Principle stresses, σ_1, σ_2, σ_3
- Von Mises Effective stress,
- Stress intensity, SI

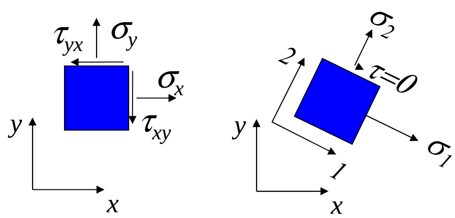
Principle Stresses, σ_1, σ_2, σ_3
(`S1,S2,S3` in ANSYS)

- Max principle tensile stress, $\sigma_1 = max(\sigma_1, \sigma_2, \sigma_3)$
- Max princ. compressive stress $\sigma_3 = min(\sigma_1, \sigma_2, \sigma_3)$
- Find (σ_1, σ_2, σ_3) from 3 roots σ of:

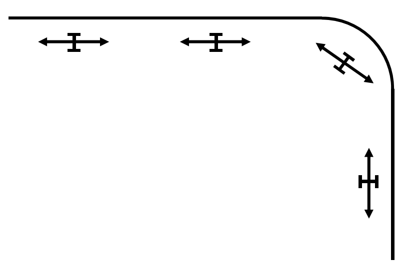
$$det\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

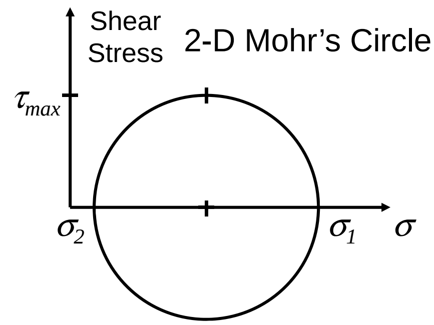
- Coordinate transformation to eliminate shear

Principal Stress Meaning



- Rotate coords until shear = 0





Stress Invariants, I_1, I_2, I_3

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \frac{1}{2}\left(\sigma_x^{\,2} + \sigma_y^{\,2} + \sigma_z^{\,2} + 2\tau_{xy}^{\,2} + 2\tau_{yz}^{\,2} + 2\tau_{zx}^{\,2} - I_1^{\,2}\right)$$

$$I_3 = det\begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix}$$

- Use to find principal stresses from 3 roots of:

$$\sigma^3 - I_1\sigma^2 - I_2\sigma - I_3 = 0$$

Stress Intensity, SI
(`SINT` in ANSYS)

$$\text{SI} = max\left(\left|\sigma_1 - \sigma_2\right|, \left|\sigma_2 - \sigma_3\right|, \left|\sigma_3 - \sigma_1\right|\right)$$

Von Mises Effective Stress, $\bar{\sigma}$
(`SEQV` in ANSYS)

always > 0

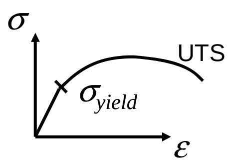
$$\bar{\sigma} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + \left(\sigma_z - \sigma_x\right)^2 + 6\left(\tau_{xy}^{\ 2} + \tau_{yz}^{\ 2} + \tau_{zx}^{\ 2}\right)}$$

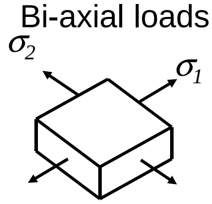
$$= \frac{1}{\sqrt{2}}\sqrt{S_1^{\ 2} + S_2^{\ 2} + S_3^{\ 2}} \quad \text{where} \quad S_1 = \frac{1}{3}\left(2\sigma_1 - \sigma_2 - \sigma_3\right)$$

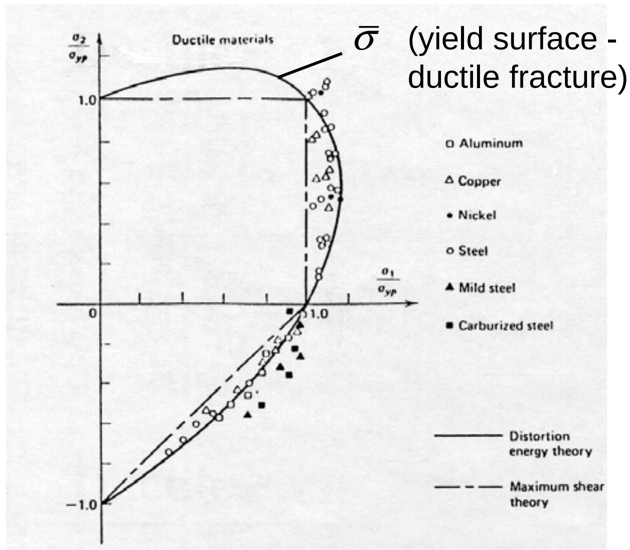
Ductile Failure

Uniaxial loads



Bi-axial loads





Von Mises Yield Criterion

Yield surface (2-D)

Also applies to polymers

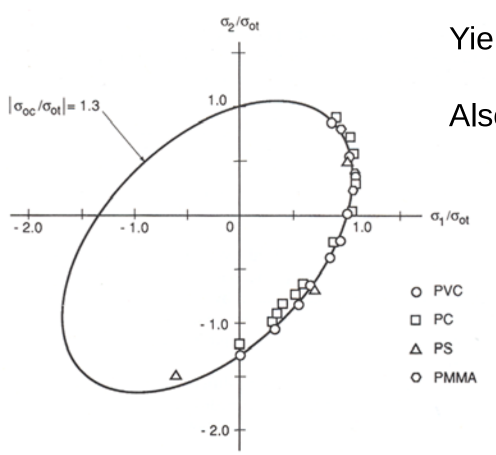


Figure 7.12 Biaxial yield data for various polymers compared to a modified octahedral shear stress theory. (After Raghava, Caddell, and Yeh [Raghava 73]; used with permission.)

Yield surface (3D)

For analylsis of complex, 3-D, stress states

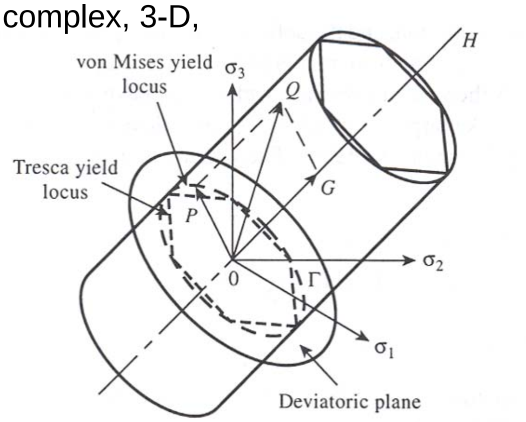
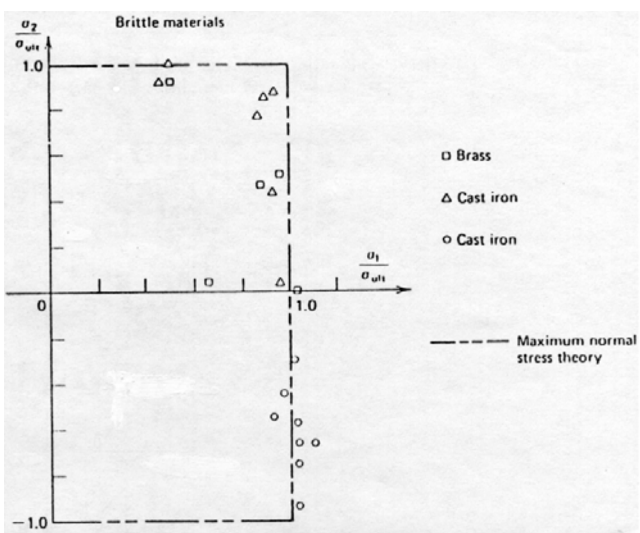


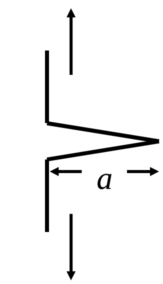
FIGURE 1.5. Geometrical representation of yield criteria in the principal stress space.

From, J. Chakrabarty, Applied Plasticity, Springer, 2000, pp.12-13.

Brittle Failure



Results Analysis



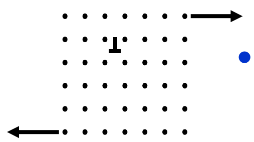
- Brittle failure: $\sigma_1 > \sigma_{max-B}$ (max principle tensile stress)

$$\sigma_{max-B} > \frac{K_{Ic}}{Y\sqrt{\pi a}}$$

fracture toughness

flaw size

geometry factor



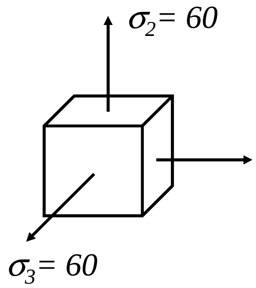
- Ductile failure: $\bar{\sigma} > \sigma_{max-D}$ (Von Mises)

$$\sigma_{max-D} = \sigma_{Yield}$$

$\sigma_{yield} \Rightarrow$ yielding occurs

$\sigma_{UTS} \Rightarrow$ ductile fracture

Example: Tri-axial Stress Condition



$\sigma_{max\text{-}D} = 40\ MPa$

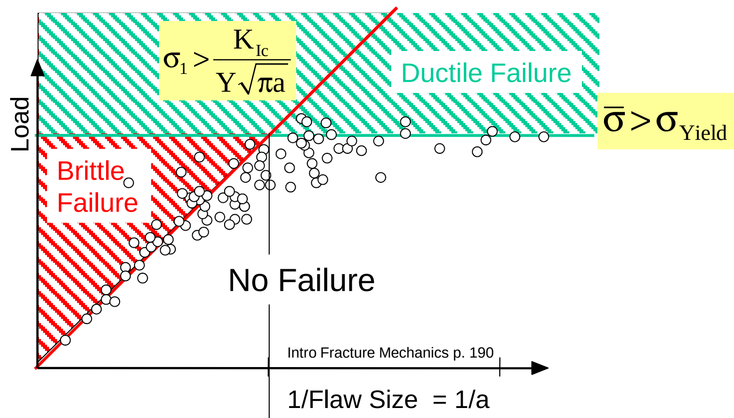
$\sigma_{max\text{-}B} = 55\ MPa$

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\sqrt{(60-60)^2 + (60-60)^2 + (60-60)^2}$$

$= 0 \quad \Rightarrow$ Ductile Failure Impossible

But BRITTLE failure is likely

Combined Failure Criteria: Yielding and Cracking



Failure controlled by large flaws

$\sigma_1 > \sigma_{max-B}$

Failure controlled by yield strength

$\bar{\sigma} > \sigma_{max-D}$

Steps in Finite Element Analysis

1. Define the real-world problem
2. Transform to a mathematical problem
   – governing PDE(s), BCs, and domain
3. Solve using FEM
4. Check
   – analytical solutions and hand calculations
   – experiments
5. Parametric studies
6. Evaluation - recommendations

